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A convenient criterion under which \mathbb{Z}_2 -graded operators are Hamiltonian

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Abstract. We formulate a simple and convenient criterion under which skew-adjoint \mathbb{Z}_2 -graded total differential operators are Hamiltonian, provided that their images are closed under commutation in the Lie algebras of evolutionary vector fields on the infinite jet spaces for vector bundles over smooth manifolds.

In this short note we consider Hamiltonian differential operators that induce the Poisson brackets on the spaces of Hamiltonian functionals on the infinite jet spaces for \mathbb{Z}_2 -graded vector bundles over smooth (super-)manifolds. In other words, we study the structures that are related to the bundles in which the fibres are split, as vector spaces, in the even and the odd components so that, in particular, the even components of the local sections commute with everything whereas the odd components of the sections anti-commute between themselves. In addition, the base of the bundles can be a supermanifold itself, whence the super-derivatives emerge; we always assume that the operators at hand are polynomial in the (super-)derivatives.

We extend a very simple criterion ([1] and [2, p. 130]), under which linear differential operators are Hamiltonian, to the \mathbb{Z}_2 -graded setup. This will be helpful, in particular, in the study of supersymmetric integrable systems. Obviously, the same criterion allows us to check the compatibility [3] of two given \mathbb{Z}_2 -graded Hamiltonian operators A_1, A_2 by verifying that the linear combinations $A_1 + \lambda A_2$ remain Hamiltonian at all λ . The tool which we elaborate is very practical and efficient: indeed, its “hardest” component amounts to the calculation of the commutator of two evolutionary vector fields (c.f. [4]). It is important that the procedure is purely algorithmic and is applicable immediately without any further adaptations (handled, e.g., by the software [5]). We recall that other methods for checking whether a given operator is Hamiltonian are available from the literature: e.g. one can re-derive the algorithmic verification procedure from [6] in the \mathbb{Z}_2 -graded setup; that concept is based on the use of variational polyvectors which are already endowed with their own grading. We finally recall that the book [7] contains another step-by-step verification procedure but (especially in the \mathbb{Z}_2 -graded case) in practice it is much more involved.

This note is structured as follows. We first extend the criterion of [1] to bosonic super-fields and super-operators, see (5). Theorem 1 in section 2 is our main result that covers the general

setup of \mathbb{Z}_2 -graded fields. Its proof, which is given here in full detail, is considerably simplified with respect to the one in [1].

All notions and constructions from the geometry of differential equations are standard ([7] and [2, 8]). We follow the notation of [1] which agrees with that of [2] but initially covered only the non-graded case. In the sequel, everything is real and C^∞ -smooth.

1. Bosonic super-fields and Hamiltonian super-operators

Let $B^n \ni x = (x^1, \dots, x^n)$ be an n -dimensional orientable manifold and let $\pi: E^{m+n} \xrightarrow{F^m} B^n$ be a vector bundle over B^n with m -dimensional fibres $F^m \ni u = (u^1, \dots, u^m)$. By $J^\infty(\pi)$ we denote the infinite jet space over π . We denote by u_σ , $|\sigma| \geq 0$, its fibre coordinates. We also denote by \mathfrak{g} the Lie algebra of evolutionary vector fields ∂_φ on $J^\infty(\pi)$ and by Ω the linear space of variational covectors, which contains the variational derivatives $\delta\mathcal{H}/\delta u$ of the Hamiltonian functionals $\mathcal{H} \in \bar{H}$ and which is dual to \mathfrak{g} with respect to the coupling \langle, \rangle that takes values in \bar{H} .

Let $A: \Omega \rightarrow \mathfrak{g}$ be a total differential operator the image of which is closed with respect to the commutation in \mathfrak{g} ,

$$[\text{im } A, \text{im } A] \subseteq \text{im } A. \quad (1)$$

This is indeed so for \mathbb{Z}_2 -graded Hamiltonian operators; the criterion in Theorem 1, see below, makes it clear that condition (1) is not superfluous for their definition. Further examples of *non-Hamiltonian* differential operators, the images of which in the Lie algebras of evolutionary vector fields are subject to the collective commutation closure but the domains of which are different from Ω , are studied in [1, 9] and [10].

The operator A transfers the Lie algebra structure $[\cdot, \cdot]_{\text{im } A}$ to the skew-symmetric bracket $[\cdot, \cdot]_A$ in the quotient $\text{dom } A / \ker A$,

$$[A(\mathbf{p}), A(\mathbf{q})] = A([\mathbf{p}, \mathbf{q}]_A), \quad \mathbf{p}, \mathbf{q} \in \Omega. \quad (2)$$

By the Leibnitz rule, two sets of summands appear in the bracket $[\partial_{A(\mathbf{p})}, \partial_{A(\mathbf{q})}] = \partial_{[A(\mathbf{p}), A(\mathbf{q})]}$ of evolutionary vector fields $\partial_{A(\mathbf{p})}$ and $\partial_{A(\mathbf{q})}$:

$$[A(\mathbf{p}), A(\mathbf{q})] = A(\partial_{A(\mathbf{p})}(\mathbf{q}) - \partial_{A(\mathbf{q})}(\mathbf{p})) + (\partial_{A(\mathbf{p})}(A)(\mathbf{q}) - \partial_{A(\mathbf{q})}(A)(\mathbf{p})). \quad (3)$$

In the first term we have used the permutability of evolutionary derivations, which are of the form $\partial_\varphi = \varphi \frac{\partial}{\partial u} + \frac{d}{dx}(\varphi) \frac{\partial}{\partial u_x} + \dots$, and total derivatives. The second term hits the image of A by construction. Consequently, the Lie algebra structure $[\cdot, \cdot]_A$ on the domain of A equals

$$[\mathbf{p}, \mathbf{q}]_A = \partial_{A(\mathbf{p})}(\mathbf{q}) - \partial_{A(\mathbf{q})}(\mathbf{p}) + \{\{\mathbf{p}, \mathbf{q}\}\}_A. \quad (4)$$

Example 1. The second Hamiltonian operator for the Korteweg–de Vries equation is $A = -\frac{1}{2} \frac{d^3}{dx^3} + u \frac{d}{dx} + \frac{d}{dx} \circ u$, where $\frac{d}{dx} = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \dots$. The image of A is closed under commutation, and the Lie algebra structure $[\cdot, \cdot]_A$ on its domain is related by the homomorphisms $\delta/\delta u$ and A to the Lie algebra $(\bar{H}, \{\cdot, \cdot\}_A)$ of Hamiltonians, endowed with the Poisson bracket, and to the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ of evolutionary vector fields, respectively (see [11]). It can easily be checked [12] that, for the above operator, the bracket $[\cdot, \cdot]_A$ on the domain Ω of A is $[p, q]_A = \partial_{A(p)}(q) - \partial_{A(q)}(p) + \frac{d}{dx}(p) \cdot q - p \cdot \frac{d}{dx}(q)$, here $p, q \in \Omega$.

The bracket $\{\{\cdot, \cdot\}\}_A$ for Hamiltonian operators A can be obtained explicitly from the Jacobi identity $[[\mathbf{A}, \mathbf{A}]] = 0$ for the Lie algebra $(\bar{H}, \{\cdot, \cdot\}_A)$ of the Hamiltonian functionals endowed by A with the Poisson bracket $\{\cdot, \cdot\}_A$; here \mathbf{A} is the representation of A by the variational Poisson bivector and $[[\cdot, \cdot]]$ is the variational Schouten bracket, see [1, 2, 4]. We now write the result of such

a derivation in local coordinates but in a properly ordered way which is slightly different from Eq. (5) in [1]: For a Hamiltonian operator $A = \|A_{\tau}^{ij} \cdot \frac{d|\tau|}{dx^{\tau}}\|$, the k -th ($1 \leq k \leq m$) component of $\{\{, \}\}_A$ equals

$$\{\{p, q\}\}_A^k = \sum_{|\sigma|, |\tau| \geq 0} \sum_{i, j=1}^m \left(\frac{d|\sigma|}{dx^{\sigma}} \right)^{\dagger} \left[q_i \cdot \frac{\partial A_{\tau}^{ij}}{\partial u_{\sigma}^k} \cdot \frac{d|\tau|}{dx^{\tau}}(p_j) \right], \quad (5)$$

where \dagger denotes the adjoint. The benefit of this notation is that formula (5) covers the super-setup of bosonic super-fields and parity-preserving Hamiltonian operators that endow the spaces of bosonic functionals with Poisson brackets. Here the multi-indices σ and τ can run through the super-derivations as well, and the partial derivatives $\partial/\partial u_{\sigma}^k$ in (5) act according to the graded Leibniz rule.

Example 2. Let $u = u_0(x, t) \cdot \mathbf{1} + \theta_1 \cdot u_1(x, t) + \theta_2 \cdot u_2(x, t) + \theta_1 \theta_2 \cdot u_{12}(x, t)$ be a scalar bosonic super-field, that is, a mapping of $\mathbb{R}^2 \ni (x, t)$ to the four-dimensional Grassmann algebra generated over \mathbb{R} by θ_1 and θ_2 satisfying $\theta_i \theta_j = -\theta_j \theta_i$. By definition, put $\mathcal{D}_i = \partial/\partial \theta_i + \theta_i \cdot d/dx$, here $1 \leq i, j \leq 2$ and it is readily seen that $\mathcal{D}_i \mathcal{D}_j + \mathcal{D}_j \mathcal{D}_i = 2\delta_{ij} \cdot d/dx$.

Consider the super-operator A_2 that comes from the N=2 classical super-conformal algebra [13] and yields the second Hamiltonian structure for the triplet of integrable N=2 supersymmetric Korteweg-de Vries equations ([14], see also [15])

$$A_2 = \mathcal{D}_1 \mathcal{D}_2 \frac{d}{dx} + 2u \frac{d}{dx} - \mathcal{D}_1(u) \mathcal{D}_1 - \mathcal{D}_2(u) \mathcal{D}_2 + 2u_x. \quad (6)$$

Let the bosonic super-sections $p, q \in \Omega$ be two arguments of A_2 . Then formula (5) yields their skew-symmetric bracket

$$\{\{p, q\}\}_{A_2} = 2\left(\frac{d}{dx} p \cdot q - p \cdot \frac{d}{dx} q\right) - \mathcal{D}_1(p) \cdot \mathcal{D}_1(q) - \mathcal{D}_2(p) \cdot \mathcal{D}_2(q), \quad (7)$$

and the validity of (4) confirms that the super-operator A_2 is indeed Hamiltonian.

2. \mathbb{Z}_2 -graded fields and the Hamiltonianity criterion

The purely bosonic setup of [1, 2] and the N=2 supersymmetry invariance in Example 2 are particular cases in the general \mathbb{Z}_2 -graded framework of $(m_0 | m_1)$ -dimensional fibre bundles π and parity-preserving Hamiltonian operators $A: \Omega \rightarrow \mathfrak{g}$ for bosonic Hamiltonian functionals.

Let \langle, \rangle denote the standard coupling $\Omega \times \mathfrak{g} \rightarrow \bar{H}$ and define $\langle | \rangle$ by setting $\langle \mathbf{b} | \mathbf{d} \rangle := \langle \mathbf{d}, \mathbf{b} \rangle$. Namely, if $\mathbf{b} = (\mathbf{b}^0, \mathbf{b}^1)$ and $\mathbf{d} = (\mathbf{d}^0, \mathbf{d}^1)$ are decomposed to even and odd-graded components, then $\langle \mathbf{b}, \mathbf{d} \rangle = \mathbf{b}^0 \cdot \mathbf{d}^0 + \mathbf{b}^1 \cdot \mathbf{d}^1$ and $\langle \mathbf{b} | \mathbf{d} \rangle = \mathbf{b}^0 \cdot \mathbf{d}^0 - \mathbf{b}^1 \cdot \mathbf{d}^1$. The definition of adjoint graded operators implies $\langle \mathbf{b}, A(\mathbf{d}) \rangle = \langle \mathbf{d}, A^{\dagger}(\mathbf{b}) \rangle = \langle A^{\dagger}(\mathbf{b}) | \mathbf{d} \rangle$.

Theorem 1. A \mathbb{Z}_2 -graded parity-preserving skew-adjoint total differential operator $A: \Omega \rightarrow \mathfrak{g}$ is Hamiltonian if and only if its image is closed under commutation and, for all $p, q, r \in \Omega$, the bracket $\{\{, \}\}_A$ in (4) satisfies the equality

$$\langle A(\{\{p, q\}\}_A) | r \rangle = : \langle p, \partial_{A(r)}(A)(q) \rangle :, \quad (8)$$

where the normal order $::$ suggests that all derivations are thrown off $A(r)$ by the graded Green formula and the arrows indicate that first $A(r)$ is moved to the right of q , and then the operator A is pushed to the left of p by Green's formula again (this is explained in the proof below). The arising argument of the skew-adjoint operator A is the bracket $\{\{p, q\}\}_A$.

Proof. Let us expand each of the three terms of the Jacobi identity,

$$\sum_{\odot} \partial_{A(p)}(\langle q, A(r) \rangle) = 0,$$

by using the Leibniz rule. We obtain

$$\sum_{\circlearrowleft} \left[\langle \partial_{A(\mathbf{p})}(\mathbf{q}), A(\mathbf{r}) \rangle + \langle \mathbf{q}, \partial_{A(\mathbf{p})}(A)(\mathbf{r}) \rangle + \langle \mathbf{q}, A(\partial_{A(\mathbf{p})}(\mathbf{r})) \rangle \right] = 0. \quad (9)$$

Consider the third term in (9) and, by the substitution principle [7], suppose that \mathbf{r} is the variational derivative of a Hamiltonian functional, whence the linearization $\ell_{\mathbf{r}}$ is self-adjoint in the graded sense. Consequently,

$$\begin{aligned} \langle \mathbf{q}, A(\partial_{A(\mathbf{p})}(\mathbf{r})) \rangle &= -\langle A(\mathbf{q}) | \partial_{A(\mathbf{p})}(\mathbf{r}) \rangle = -\langle A(\mathbf{q}) | \ell_{\mathbf{r}}(A(\mathbf{p})) \rangle = -\langle A(\mathbf{p}) | \ell_{\mathbf{r}}^{\dagger}(A(\mathbf{q})) \rangle \\ &= -\langle A(\mathbf{p}) | \ell_{\mathbf{r}}(A(\mathbf{q})) \rangle = -\langle \ell_{\mathbf{r}}(A(\mathbf{q})), A(\mathbf{p}) \rangle = -\langle \partial_{A(\mathbf{q})}(\mathbf{r}), A(\mathbf{p}) \rangle. \end{aligned}$$

Substituting this back in (9) and taking the sum over the cyclic permutations, we cancel 3×2 terms, except for

$$\langle \mathbf{q}, \partial_{A(\mathbf{p})}(A)(\mathbf{r}) \rangle + \langle \mathbf{r}, \partial_{A(\mathbf{q})}(A)(\mathbf{p}) \rangle + \langle \mathbf{p}, \partial_{A(\mathbf{r})}(A)(\mathbf{q}) \rangle = 0. \quad (10)$$

Now we consider separately the first and second summands in (10), paying due attention to the order of graded objects and the directions the derivations act in. First, applying the even vector field $\partial_{A(\mathbf{p})}$ to the equality $\langle \mathbf{q}, A(\mathbf{r}) \rangle = \langle A^{\dagger}(\mathbf{q}) | \mathbf{r} \rangle$ and using $A^{\dagger} = -A$, we conclude that

$$\langle \mathbf{q}, \partial_{A(\mathbf{p})}(A)(\mathbf{r}) \rangle = -\langle \partial_{A(\mathbf{p})}(A)(\mathbf{q}) | \mathbf{r} \rangle.$$

Likewise, the second summand in (10) gives

$$\langle \mathbf{r}, \partial_{A(\mathbf{q})}(A)(\mathbf{p}) \rangle = \langle \partial_{A(\mathbf{q})}(A)(\mathbf{p}) | \mathbf{r} \rangle.$$

Hence from (10) we obtain

$$\langle \partial_{A(\mathbf{p})}(A)(\mathbf{q}) | \mathbf{r} \rangle - \langle \partial_{A(\mathbf{q})}(A)(\mathbf{p}) | \mathbf{r} \rangle = \langle \mathbf{p}, \partial_{A(\mathbf{r})}(A)(\mathbf{q}) \rangle.$$

Integrating the right-hand side by parts, we move the skew-adjoint operator A off \mathbf{r} and obtain the bracket $\{\{\mathbf{p}, \mathbf{q}\}\}_A$ as its argument.

We have shown that if the bracket induced on the domain of a given graded skew-adjoint operator A with involutive image, see (4), coincides with the bracket $\{\{, \}\}_A$ emerging from (8), then A is indeed Hamiltonian, and *vice versa*. This concludes the proof.

Example 3. Writing the super-operator (6) in components (see Appendix A below), now with $p_i = \delta\mathcal{H}/\delta u_i$, whence p_0 and p_{12} are even and p_1, p_2 are odd, we obtain the (4×4) -matrix operator [13]

$$A_2 = \begin{pmatrix} -\frac{d}{dx} & -u_2 & u_1 & 2u_0 \frac{d}{dx} + 2u_{0;x} \\ -u_2 & \left(\frac{d}{dx}\right)^2 + u_{12} & -2u_0 \frac{d}{dx} - u_{0;x} & 3u_1 \frac{d}{dx} + 2u_{1;x} \\ u_1 & 2u_0 \frac{d}{dx} + u_{0;x} & \left(\frac{d}{dx}\right)^2 + u_{12} & 3u_2 \frac{d}{dx} + 2u_{2;x} \\ 2u_0 \frac{d}{dx} & -3u_1 \frac{d}{dx} - u_{1;x} & -3u_2 \frac{d}{dx} - u_{2;x} & \left(\frac{d}{dx}\right)^3 + 4u_{12} \frac{d}{dx} + 2u_{12;x} \end{pmatrix}. \quad (11)$$

The application of Theorem 1 is particularly transparent since the coefficients of (11) are linear functions. The right-hand side of (8) yields the four components of the skew-symmetric bracket $\{\{\mathbf{p}, \mathbf{q}\}\}_{A_2}$,

$$\begin{aligned} \{\{\mathbf{p}, \mathbf{q}\}\}_{A_2}^0 &= 2(p_{0;x}q_{12} - p_{12}q_{0;x}) - (p_{1;x}q_2 + p_2q_{1;x}) + (p_{2;x}q_1 + p_1q_{2;x}), \\ \{\{\mathbf{p}, \mathbf{q}\}\}_{A_2}^1 &= 2(p_{1;x}q_{12} - p_{12}q_{1;x}) + (p_0q_2 - p_2q_0) + (p_{12;x}q_1 - p_1q_{12;x}), \\ \{\{\mathbf{p}, \mathbf{q}\}\}_{A_2}^2 &= 2(p_{2;x}q_{12} - p_{12}q_{2;x}) + (p_1q_0 - p_0q_1) + (p_{12;x}q_2 - p_2q_{12;x}), \\ \{\{\mathbf{p}, \mathbf{q}\}\}_{A_2}^{12} &= 2(p_{12;x}q_{12} - p_{12}q_{12;x}) - p_1q_1 - p_2q_2. \end{aligned}$$

This is the component expansion of (7); see [15] for further results on the geometry of the $N=2$ supersymmetric $a=4$ -Korteweg-de Vries equation.

Conclusion

Theorem 1 provides an exact and exhaustive answer on the question whether a given skew-adjoint \mathbb{Z}_2 -graded differential operator with involutive image in \mathfrak{g} is Hamiltonian:

- take sections $\mathbf{p}, \mathbf{q} \in \Omega$ and calculate the commutator $[A(\mathbf{p}), A(\mathbf{q})]$, omitting the standard terms $\partial_{A(\mathbf{p})}(\mathbf{q}) - \partial_{A(\mathbf{q})}(\mathbf{p})$;
- calculate the m -tuple $A(\{\{\mathbf{p}, \mathbf{q}\}\}_A)$ by using formula (8).

If the two expressions coincide, the operator A is Hamiltonian.

Appendix A. Field–superfield correlation for variational derivatives

Given a Hamiltonian functional $\mathcal{H} = \int \mathbf{h}[\mathbf{u}] d\theta dx$, $d\theta = d\theta_1 \cdots d\theta_N$ whose density \mathbf{h} is a differential superfunction in $(m_0 | m_1)$ super-fields \mathbf{u}^α of 2^N components each, what is the correlation between the components p_I^α of the variational derivatives

$$\frac{\delta \mathcal{H}}{\delta \mathbf{u}^\alpha} = p_\emptyset^\alpha \cdot \mathbf{1} + \dots + \theta_1 \cdots \theta_N \cdot p_{(1, \dots, N)}^\alpha = \sum_{|I|=0}^N \theta_I \cdot p_I^\alpha$$

with respect to the super-fields

$$\mathbf{u}^\alpha = u_\emptyset^\alpha \cdot \mathbf{1} + \dots + \theta_1 \cdots \theta_N \cdot u_{(1, \dots, N)}^\alpha = \sum_{|J|=0}^N \theta_J \cdot u_J^\alpha \quad (\text{A.1})$$

and, on the other hand, the variational derivatives $\psi_J^\alpha = \delta \mathcal{H} / \delta u_J^\alpha$ of the functional \mathcal{H} with respect to the $(m_0 + m_1) \cdot 2^N$ components u_J^α of the super-fields (here $1 \leq \alpha \leq m_0 + m_1$)? We note that the answer to this question (for which it suffices to consider only one super-field, hence we shall omit the superscripts α) also encodes the Hamiltonian super-operators in their matrix component form (e.g., see (11)).

Proposition. Let \mathbf{u} be an $N \geq 1$ super-field (A.1) and $\mathcal{H} = \int \mathbf{h}[\mathbf{u}] d\theta dx$ be a Hamiltonian super-functional. For all multi-indexes J of length $|J|$ such that $0 \leq |J| \leq N$, denote by I the multiindex of length $|I| = N - |J|$ such that their disjoint union is $I \sqcup J = \{1, \dots, N\}$. Then the sought correlation between p_I and ψ_J is

$$\psi_J = (-1)^{(|\mathcal{H}| - |\mathbf{u}| - |I|) \cdot |J|} \cdot (-1)^{\overline{I, J}} \cdot p_I, \quad (\text{A.2})$$

where $|\mathcal{H}|$ is the parity of the Hamiltonian, $|\mathbf{u}|$ is the parity of the super-field, and the ordered concatenation of the multi-indexes $\overline{I, J}$ is a permutation of $1, \dots, N$.

Example 4. Suppose $N=2$ as in Examples 2 and 3. Then the correlation between the component expansion $\mathbf{p} = p_0 \cdot \mathbf{1} + \theta_1 p_1 + \theta_2 p_2 + \theta_1 \theta_2 p_{12}$ of the variational derivative $\delta \mathcal{H} / \delta \mathbf{u}$ and the variations $\psi_I = \delta \mathcal{H} / \delta u_I$ with respect to the components u_I , $I \in \{0, 1, 2, 12\}$, of the super-field $\mathbf{u} = u_0 \cdot \mathbf{1} + \theta_1 u_1 + \theta_2 u_2 + \theta_1 \theta_2 u_{12}$ is given by the formula¹

$$\psi_0 = p_{12}, \quad \psi_1 = p_2, \quad \psi_2 = -p_1, \quad \text{and} \quad \psi_{12} = p_0. \quad (\text{A.2}')$$

We thus recover the (4×4) -matrix operator (11) by writing in components both \mathbf{u} and the argument \mathbf{p} of the Hamiltonian super-operator (6), by inserting the correlation (??') for the

¹ Although formula (??') is valid for all \mathcal{H} , it is particularly transparent for $\mathbf{h} = \frac{1}{2} \mathbf{u}^2$ such that the identity $\frac{1}{2} \int \mathbf{u}^2 d\theta dx = \int (u_0 u_{12} - u_1 u_2) dx$ yields $\mathbf{p} = \mathbf{u}$ and $\vec{\psi} = {}^t(u_{12}, u_2, -u_1, u_0)$, the superscript t denoting the transposition.

components of \mathbf{p} , and then reordering the columns of the matrix operator A_2 so that its argument $\vec{\psi}$ acquires the standard form ${}^t(\psi_0, \psi_1, \psi_2, \psi_{12})$. To let the notation of Example 3 match Theorem 1, we re-denote by p_i the variational derivatives $\delta\mathcal{H}/\delta u_i$.

Proof of Proposition. Consider the Hamiltonian $\mathcal{H} = \int \mathbf{h} d\theta dx$. Varying the super-field \mathbf{u} by $\delta\mathbf{u}$, we throw all the derivatives off $\delta\mathbf{u}$ using multiple integration by parts in x , which yields $\int \mathbf{p} \cdot \delta\mathbf{u} d\theta$. Next, let us insert the expansions $\mathbf{p} = \sum_I \theta_{i_1} \cdots \theta_{i_{|I|}} \cdot p_I$ and $\delta\mathbf{u} = \sum_J \theta_{j_1} \cdots \theta_{j_{|J|}} \cdot u_J$ in this super-integral. By its definition, only the coefficient of $\theta_1 \cdots \theta_N$ in the product $\mathbf{p} \cdot \delta\mathbf{u}$ contributes to the integral's value, hence only the complementary multi-indexes $I \sqcup J = \{1, \dots, N\}$ count. Pushing p_I through θ^J , we accumulate the sign $(-1)^{|p_I| \cdot |J|}$, where $|p_I| = |\mathcal{H}| - |\mathbf{u}| - |I|$. Finally, reordering the product $\theta_I \cdot \theta_J$ to $\theta_1 \cdots \theta_N$, we obtain the sign of the permutation $\overline{I}, \overline{J}$.

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